Kazhdan-Lusztig Theory

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1 Verma Module Characters

Recall last week we introduced the Category $\mathcal{O}$ for a Lie algebra and the Verma module simple quotients $M(\lambda) \to L(\lambda)$, $\lambda \in \mathfrak{h}^*$ a weight, were exactly the simple objects of $\mathcal{O}$. When these quotients were finite-dimensional (i.e. λ dominant integral) we had the Weyl Character Formula

Theorem 1 (Weyl Character)**.** *For $L(\lambda)$ finite-dimensional,*

$$Ch(L(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0}} \quad \textit{(Note } w \cdot \rho = w(\lambda + \rho) - \rho\textit{)}$$

for W the Weyl group, ρ the half-sum of positive roots, and formal character

$$Ch(M) = \sum_{\lambda \in \mathfrak{h}^*} dim(M_\lambda) e^\lambda$$

We wanted to compute the composition factors of other Verma modules and we had obtained that using translation functors we can reduce the computation of characters of simple modules to computations in $\mathcal{O}_0$, the subcategory where $\mathcal{Z}(\mathfrak{g})$ acts nilpotently. The specific formula we had was that in the Grothendieck group $K_0(0)$ (since characters are additive on exact sequences, it's equivalent to look at the Grothendieck group),

$$[M(\lambda)] = \sum_{\mu = w \cdot \lambda} [M(\lambda) : L(\mu)][L(\mu)]$$

$$\implies [L(\lambda)] = \sum_{y \leq w} b_{y,w} [M((yw^{-1}) \cdot \lambda)] = \sum_{x \leq w} a_{x,w} [M_x], \quad b_{y,w} \in \mathbb{Z}$$

Kazhdan-Lusztig Theory gives a way to compute these $a_{x,w}$ as some $P_{x,w}(1)$ for $P_{x,w} \in \mathbb{Z}[q]$.

Example 1. Let's recall what happens for $\mathfrak{sl}_2 = \langle e, f, h \rangle$ with $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. The finite-dimensional irreducibles are all found in the representation on functions (where e, f, h act concretely by $e = x\frac{\partial}{\partial y}, f = y\frac{\partial}{\partial x}, h = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$)

$$\text{Fun}(\mathbb{C}^2) = \bigoplus_{n \geq 0} L_n \boxtimes \mathbb{C}(x, y), \;\; L_n = \text{span}\{x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n\}$$

The computation for the character of L_n directly is just (the weight spaces are 1 dimensional and go from n to $-n$ stepping by two):

$$\mathrm{Ch}(L_n) = e^n + e^{n-2} + \cdots + e^{-n+2} + e^{-n} = \frac{e^n}{1-e^{-2}} - \frac{e^{-n-2}}{1-e^{-2}} =$$

$$= \mathrm{Ch}(M_n) - \mathrm{Ch}(M_{-n-2}) = \mathrm{Ch}(M(\lambda_n)) - \mathrm{Ch}(M(s \cdot \lambda_n))$$

which is the Weyl character formula form.

Notice that these $L_n = \Gamma(\mathbb{P}^1, \mathcal{O}(n))$ (recall this is "bundle with transition functions between the two charts given by $(\frac{x_j}{x_i})^n$"). We will discuss this more later.

2 Combinatorial Picture

A Coxeter group is a group generated by reflections (order 2 elements) $\{s_1, \ldots, s_n\}$ with only relations of the form $(s_i s_j)^{m_{i,j}} = 1$. Weyl groups are a particular example. Explicitly, W is the quotient of the normalizer $N_G(T)/T$ for T a maximal torus in a Borel subgroup, or also W is the group generated (say, in the orthogonal group) by reflections across lines perpendicular to the simple roots of G. For GL_n or $\mathfrak{sl}_n$ the Weyl group is just the symmetric group on n letters.

Let W be the Weyl group (or any Coxeter group) and S the set of generating reflections. For S_n this is the set $\{(12), (23), \ldots, (n-1\ n)\}$ of simple transpositions. The length of a Weyl group element is the fewest number of simple reflections in an expression for the element and elements are ordered $y \leq w$ means $w = y s_1 \cdots s_k$ with each right multiplication increasing the length.

Definition 1. The **Hecke algebra $\mathcal{H}_W$ of W** is the free $\mathbb{Z}[q^{\pm 1/2}]$-module (we'll see why we need square roots) on $\{T_w\}_{w \in W}$ with identity T_1 and multiplication via

$$\begin{cases} T_s T_w = T_{sw} \text{ for } \ell(sw) > \ell(w) \\ T_s T_w = (q-1) T_w + q T_{sw} \text{ for } \ell(sw) < \ell(w) \end{cases} \tag{1}$$

Notice that if we set $q = 1$ both relations collapse to

$$T_s T_w = T_{sw}$$

, so the Hecke algebra is a deformation of the integral group algebra $\mathbb{Z}[W]$ of W.

However, when we quantize by q even though $s^2 = 1$ we have

$$\boxed{T_s^2 = (q-1)T_s + qT_1}$$

We can still make T_s invertible because

$$T_s\left(aT_1 + bT_s\right) = aT_s + b\left((q-1)T_s + qT_1\right) = (a + b(q-1))T_s + bqT_1$$

$$\implies \text{ by setting } b = q^{-1}, a = q^{-1} - 1 \text{ we have } T_s\left((1-q^{-1})T_1 + q^{-1}T_s\right) = T_1.$$

Together with

$$((q^{-1}-1)T_1 + q^{-1}T_s)T_s = q^{-1}T_s - T_s + q^{-1}(q-1)T_s + q^{-1}qT_1 = T_1$$

we have that $\boxed{T_s^{-1} = (q^{-1}-1)T_1 + q^{-1}T_s}$.

Since every element w of the Weyl group is generated by a (nonunique) minimal string of generators in S, $w = s_1 \cdots s_n$, we have

$$T_w = T_{s_1\cdots s_n} = T_{s_1}\cdots T_{s_n} \implies T_w^{-1} = T_{s_n}^{-1}\cdots T_{s_1}^{-1} \implies \boxed{\text{all } T_w \text{ are units}}$$

An inductive argument shows the formula for T_s^{-1} generalizes as

$$(T_{w^{-1}})^{-1} = (-1)^{\ell(w)}q^{-\ell(w)}\sum_{y\leq w}(-1)^{\ell(y)}R_{y,w}(q)T_y, \quad R_{y,w}\in\mathbb{Z}[q] \text{ of degree } \ell(w)-\ell(y)$$

Definition 2. The **bar involution** on $\mathcal{H}_W$ is given by

$$\begin{cases} \bar{q} = q^{-1} \\ \overline{T_w} = (T_{w^{-1}})^{-1} \end{cases} \tag{2}$$

Can we find a basis of $\mathcal{H}_W$ indexed by W still and stable under the involution? For s a simple reflection

$$\overline{A(q)T_1 + B(q)T_s} = A(q^{-1})T_1 + B(q^{-1})((q^{-1}-1)T_1 + q^{-1}T_s) =$$

$$= \Big(A(q^{-1}) + B(q^{-1})(q^{-1}-1)\Big)T_1 + q^{-1}B(q^{-1})T_s$$

So for self-duality we want

$$B(q) = B(q^{-1})q^{-1} \quad \text{and} \quad A(q)-A(q^{-1}) = B(q^{-1})(q^{-1}-1) = q^{1/2}(q^{-1}-1) = (-q^{1/2})-(-q^{-1/2})$$

$$\implies B(q) = q^{-1/2}, A(q) = -q^{1/2}$$

$$\implies \boxed{C_s = q^{-1/2}(T_s - qT_1)} \text{ or } \boxed{C_s = -q^{-1/2}T_1 + q^{-1/2}T_s} \text{(more geometric form)}$$

Notice that we needed square roots of q to make this calculation work. The nicest coefficients for an invariant basis we can get will be of the following form:

Theorem 2 (due to Kazhdan-Lusztig)**.** *There is a* ***unique*** *basis $\{C_w\}_{w\in W}$ of $\mathcal{H}_W$ fixed under the involution subject to the normalization:*

1. $C_w = (-1)^{\ell(w)}q^{\ell(w)/2}\sum_{y\leq w}(-1)^{\ell(y)}q^{-\ell(y)}\overline{P_{y,w}(q)}T_y$ *for* $\boxed{P_{y,w}\in\mathbb{Z}[q]}$
2. $P_{w,w} = 1$ *and deg* $P_{y,w}(q) \leq \frac{\ell(w)-\ell(y)-1}{2}$

Example 2.
- For S_2, $C_1 = T_1$ and

$$C_s = q^{-1/2}(T_s - qT_1) = -q^{1/2}(T_1 + (-1)q^{-1}\cdot\bar{1}\cdot T_s)$$

$$\implies P_{1,s} = 1$$

- For S_3, the group elements are $1, s_1 = (12), s_2 = (23), s_1s_2, s_2s_1, s_1s_2s_1$. The degree bound says the biggest degree we could have is $\frac{3-0-1}{2} = 1$ if y trivial and otherwise degree 0. But $P_{1,w} = 1$ because

$$\overline{T_1} = T_1 \implies (-1)^0 q^{-0} P_{1,w}(q^{-1}) = (-1)^0 q^0 P_{1,w}(q)$$

$$\implies P_{1,w}(q^{-1}) = P_{1,w}(q) \implies P_{1,w}(q) = \text{constant}$$

  So in fact all polynomials are 1 for S_3.

- For S_4, will get the first nontrivial polynomials which are each $P_{y,w}(q) = q + 1$ when nontrivial. One of the nontrivial ones is $P_{t,tsut}$ for $S_4 = \langle s, t, u \rangle$ which does have degree bounded by $\frac{4-1-1}{2} = 1$.

- **Theorem due to Polo [1999]**: Any polynomial of degree d with integer nonnegative coefficients is the Kazhdan-Lusztig polynomial of some pair y, w in in the symmetric group of order $1 + d + P(1)$. So K-L polynomials are arbitrarily bad.

Proof Idea. Induct on the length of w. If $w = w's$, then try just multiplying together the $C_{w'}, C_s$ and then do a series of corrections to get better expressions. □

The Kazhdan-Lusztig Conjecture is that if we evaluate back at $q = 1$ (the same value where the Hecke algebra became the group algebra) we get the coefficients in our multiplicity formula.

Theorem 3 (Kazhdan-Lusztig "Conjecture")**.** *In the principal block $\mathcal{O}_0$ and simple quotient Verma module $M_w \twoheadrightarrow L_w$ where $M_w = M(w \cdot (-2\rho)), L_w = L(w \cdot (-2\rho))$ (all of the simple modules in $\mathcal{O}_0$) for $w \in W$ Weyl group,*

$$[L_w] = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)[M_y]$$

3 Geometry

Where do the Hecke algebras come from geometrically? They are supposed to tell you about the geometry of Schubert varieties (their intersection cohomology). Recall that our original setup was we take our group G, say $G = GL_2$, and a Borel subgroup B, say the subgroup of upper triangular matrices, T a maximal torus in G:

$$G = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

The Weyl group is then $N_G(T)/T$ (in this case $N_G(T)$ is permutation matrices but with any nonzero entries in the 1 spots, so can be written as DP for D diagonal, P permutation. Which is why Weyl group is S_2). The flag variety is then G/B. For GL_2 this is the space of lines in the plane because G acts transitively on the lines with stabilizer B. Finally we had the Bruhat decomposition

$$G = \bigsqcup_{w \in W} B\tilde{w}B$$

(think of this as any two Borels contain a common torus, which is just a basis, ordered in possibly two different ways, giving the permutation). The **Schubert cells** B_w are the image in the flag variety of BwB and their closures are the **Schubert varieties** X_w, each cell is isomorphic to $\mathbb{A}^{\ell(w)}$. The Weyl group ordering tells us geometrically about the inclusions, a stratification by affine opens:

$$X_w = \bigcup_{y \leq w} B_y$$

(and $B_y \subseteq X_w$ exactly when $y \leq w$). For GL_2 this is just the point and inclusion into $\mathbb{P}^1$. The first geometric correspondence is that

$$\boxed{T_w \sim j_!(\underline{\mathbb{C}}_{B_w})}$$

the shriek extension (extension by 0) of the constant sheaf.

Recall that the way we got these q-deformations was looking at the Frobenius over nonzero characteristic. This map is a canonical endomorphism on varieties that on $\mathbb{P}^1$ is given on closed points by $[x : y] \mapsto [x^p : y^p]$. What does this do on cohomology? If we use etale cohomology (roughly, "agrees with standard cohomology for the variety over $\mathbb{C}$") then we can lift the endomorphism to and endomorphism on cohomology. For $\mathbb{P}^1$ the nonzero cohomology degrees are $H^0(\mathbb{P}^1) = \mathbb{C}$ and $H^2(\mathbb{P}^1) = \mathbb{C}$. The Frobenius acts trivially on 0th cohomology (since it's canonically defined and acts trivially on a point). On top cohomology,

$$[x : 1] \mapsto [x^p : 1] \implies \deg(Fr) = p \implies \text{acts by } p$$

Similarly using something like Kunneth we have that $H^*(\mathbb{P}^1 \times \mathbb{P}^1)$ is

$$H^0 = \mathbb{C} \quad H^1 = 0 \quad H^2 = \mathbb{C} \oplus \mathbb{C} \text{ acts by } p \quad H^3 = 0 \quad H^4 = \mathbb{C} \otimes \mathbb{C} \text{ acts by } p^2$$

Therefore, roughly we expect Frobenius acts on ith cohomology by $q^{i/2}$ (I'll now use q instead of p).

To make this actually work we need to instead use intersection cohomology because our Schubert varieties will have singularities, which corresponds to using $(R\Gamma(\underline{\mathbb{C}}_X))$ global sections of a sheaf called the IC (intersection cohomology) sheaf instead of the constant sheaf (so $R\Gamma(IC_X)$). In particular, in the category of B-constructible perverse sheaves on G/B, the IC_w are the simple objects. Perverse sheaves lived in a derived category so these aren't sheaves per se but complexes of sheaves.

Without giving an exact definition of what the IC sheaf is, I'll draw a picture of what it does. If we have some singularity (draw two glued together spheres) then the stalks away from the singularity should be just the constant sheaf $\underline{\mathbb{C}}$. But at the singularity we should have stalk $\mathbb{C} \oplus \mathbb{C}$. So it's

$$IC_{doubleloop} = \underline{\mathbb{C}}_{left} \oplus \underline{\mathbb{C}}_{right}$$

This gives our next geometric correspondence

$$\boxed{C_w \sim IC_{X_w}}$$

This lines up with our understanding of $\mathbb{P}^1$ because it's just the direct sum

$$\boxed{IC_s \sim \delta_e + \delta_s \iff C_s \sim T_e + T_s}$$

Where does the bar involution come from? Recall that for sufficiently nice manifolds Poincare duality gives an isomorphism between ith cohomology and $n-i$th cohomology for n the top degree. This generalizes to intersection cohomology for IC sheafs and the map we get is the bar involution

$$\boxed{\text{bar involution} \sim \text{Verdier duality}}$$

The precise statement is that you can form the character of a perverse sheaf $\mathcal{F}$ on the flag variety of roughly the form

$$\text{Ch}(\mathcal{F}) = \sum_{w \in W, k \in \mathbb{Z}} (\dim H^{-\ell(w)-k}) q^k \delta_w$$

and under the bar involution for sufficiently nice $\mathcal{F}$,

$$\boxed{\text{Ch}(\mathbb{D}\mathcal{F}) = \overline{\text{Ch}\mathcal{F}}}$$

so our character formula is self-dual. As a side note, the factors of $q^{-1/2}$ we get come from re-indexing around middle cohomology.

For a general IC sheaf, Frobenius will act on degree $2i$ by q^i, and so if we label the rank in each degree as $\underline{\mathbb{C}}^{a_{2i}}$, then when we take trace of the Frobenius operator we get a polynomial

$$\boxed{Tr(Fr) = 1 + a_1 q + a_2 q^2 + \cdots \sim \text{Kazhdan-Lusztig Polynomial}}$$

In particular the $P_{y,w}$ will come from the trace of the Frobenius on $i^*_{ByB} IC_{\overline{BwB}}$. The precise statement of how the Kazhdan-Lusztig polynomial tells you about singularities is that

$$\boxed{P_{y,w}(q) = q^{\ell(w)-\ell(v)} \iff X_w \text{ is rationally smooth at a generic point of } X_y}$$

Example 3. Recall that the first nontrivial K-L polynomial we get is in $S_4 = \langle s, t, u \rangle$ for K-L polynomial $P_{t,tsut}(q) = q + 1$. Geometrically this is because looking at

$$i^*_{BtB} IC_{tsut} \to \text{get constant sheaf} \underline{\mathbb{C}} \text{ away from the copy of } \mathbb{P}^1_t$$

$$\to \text{get } \underline{\mathbb{C}} \oplus \underline{\mathbb{C}[-2]} \text{ with Fr acting by } 1, q$$

$$\implies Tr(Fr) = q + 1$$