Fall 2024 MATH 33A Worksheet 5: 5.3, 5.4

Exercise 1. True or false:

(a) If A is orthogonal then it is invertible.

(b) If A is symmetric it is invertible.

(c) The entries of an orthogonal matrix are all less than or equal to 1 in absolute value.

(d) Let V be a subspace of $\mathbb{R}^n$ and B the matrix for orthogonal projection onto V. Then $B^2 = B$.

(e) If $\vec{v_1}, \ldots, \vec{v_m}$ is a basis of unit length vectors for a subspace V, there is an orthonormal basis of V containing the vectors $\vec{v_1}$ and $\vec{v_2}$.

(f) For all $\vec{v}, \vec{w} \in \mathbb{R}^n$, $|\vec{v} \cdot \vec{w}|^2 \leq ||\vec{v}||^2||\vec{w}||^2$ with equality if and only if $\vec{v}, \vec{w}$ are perpendicular.

Exercise 2. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 2 & 1 & 1 & -4 \end{bmatrix}$$

(a) Find an orthonormal basis $\mathfrak{B} = \{\vec{u}_1, \vec{u}_2\}$ for $\ker A$.

(b) Using your basis from part (a), find the matrix B for orthogonal projection onto $\ker A$.

(c) Extend $\mathfrak{B}$ to an orthonormal basis $\mathfrak{C}$ for $\mathbb{R}^4$. Find the matrix for the orthogonal projection in part (b) with respect to the basis $\mathfrak{C}$.

Exercise 3. Find all orthogonal 2×2 matrices.

Exercise 4. Let $A = \begin{bmatrix} 3 & 2 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 4 & -6 & -3 \end{bmatrix}$, $V = \text{im} A$.

(a) Find the projection matrix B for proj_V, projection onto V.

(b) Using B, determine whether $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \in \text{im} A$ (B is projection onto V, so $\vec{v}$ is in V if and only if $B \cdot \vec{v} = \vec{v}$).

(c) Find the least squares solution to $A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$.

Exercise 5. Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$